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High-temperature series analysis of an O(2)-symmetric spin model with discretely valued interaction on 3D lattices

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Abstract. A detailed analysis is given of high-temperature series expansions for the susceptibility of an O(2)-symmetric spin model with discretely valued spin-spin interaction on SC, BCC and FCC lattices. This analysis indicates that the model is in the same universality class as the regular 3D O(2) spin model. New results are given for the critical amplitudes. We also list and discuss series expansions for the free energy and specific heat.

1. Introduction

The critical properties of a statistical model are understood to depend on the space P in which the order parameter lies, the (zero-field) symmetry group G of the Hamiltonian or Euclidean action, and the dimensionality d. At a very general level, one may classify such a model according to whether P and G are discrete or continuous. In commonly studied models, the interaction between the fundamental variables, for example spins, is taken as a continuous function if these variables are continuous, and discrete if they are discrete. Examples include the Ising model and its generalisations to Z_N clock and (scalar) Potts models, vertex models and O(N) models. What happens if the variables are continuous but the interaction is discrete? In particular, if one takes a given model with continuous P, G and interaction, and changes this interaction to a discretely valued one, does this change the universality class of the model? Moreover, introducing a discretely valued interaction in a model with continuous variables yields a far-reaching property of non-zero ground-state disorder. However, this disorder is not necessarily associated with any frustration. What effect does this type of ground-state disorder have on the critical properties of the theory and on the long-range order?

These questions were, to our knowledge, first studied by Guttmann *et al* (1972), Guttmann and Joyce (1973) and Guttmann and Nymeyer (1978). These authors considered an O(2)-symmetric spin model with a (nearest-neighbour) spin-spin interaction of the form $\mathrm{sgn}(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)$. Guttmann and Joyce (1973) obtained an exact (zero-field) solution in one dimension which showed that the model remained disordered, with finite correlation length and no long-range order, for all temperatures including $T = 0$. Using also a discretised form for the coupling to an external magnetic field, namely $\mathrm{sgn}(\boldsymbol{H} \cdot \boldsymbol{S}_i)$, Guttmann and his collaborators calculated high-temperature series expansions for the (zero-field) specific heat and a certain quantity analogous to the susceptibility. From these, it was concluded that, for dimensionality $d = 3$, the model was in the same universality class as the regular O(2)-symmetric spin model with interaction $\boldsymbol{S}_i \cdot \boldsymbol{S}_j$.

Realisations of models with continuous variables but discretely valued interactions arose subsequently in several contexts. Studies were carred out (Barber *et al* 1985, Barber and Shrock 1985) of a 4D lattice gauge theory with local U(1) gauge invariance and an action containing both a continuous interaction defined on plaquettes and an integer-valued monopole density interaction defined as a function of the twelve angles parametrising the gauge group elements on the links of each 3-cube. The monopole density operator is necessarily discretely valued, since the monopole charges are integers, which, in turn, is a result of the mathematical property that the first homotopy groups of S^1 is $\pi_1(S^1) = Z$. Thus the lattice gauge theory with this monopole density operator is an example, albeit a rather complicated one, of models with continuous variables (the U(1) gauge group elements on each link) and a discretely valued interaction. Related studies (Kohring *et al* 1986, Kohring and Shrock 1987) were carried out of a 3D spin model with global O(2) symmetry and a Hamiltonian consisting of the usual $\boldsymbol{S}_i \cdot \boldsymbol{S}_j$ spin-spin interaction together with a local vortex loop density operator defined on plaquettes (in a simple cubic lattice). The vortex loop density operator is a discretely (integer) valued function of the four angles specifying the orientations of the spins on the corners of the plaquette. As in the 4D U(1) lattice gauge theory, it is a topological density and takes on discrete values as a consequence of the fact that $\pi_1(S^1) = Z$. A third example is provided by neurons in a neural network. These may be viewed as operators which depend on a number of continuously varying input voltages from synaptic connections to other neurons, and which take on two values (say 1 and 0, corresponding to firing or resting). A final example is a non-linear circuit device which, depending on continuously varying input voltages or currents, produces an output current or voltage which has two (or a discrete set of) values. It is thus clear that there is ample physical motivation for the study of models with continuous variables and discretely valued interactions, in addition to the underlying statistical mechanical issues which spring to mind.

In view of the complexity of the integer-valued twelve-angle monopole density operator in the 4D U(1) lattice gauge theory and the four-spin vortex density operator in the 3D O(2) spin model, there was a motivation for constructing a model with continuous variables having a simpler discretely valued interaction. One of the simplest such models, and hence an optimal one for study, is the 3D O(2) spin model with a nearest-neighbour two-spin interaction taking on just two discrete values. A brief report on an analysis of this model was given by Lee and Shrock (1987). In the present paper we shall present the details of this analysis, together with new calculations of the critical amplitudes. We shall also take this opportunity to compare our results with the earlier work by Guttmann and Joyce (1973) and Guttmann and Nymeyer (1978) on a closely related model. Our main tool is the analysis of high-temperature series expansions for the susceptibility in this model, for the simple cubic (SC), body-centred cubic (BCC) and face-centred cubic (FCC) lattices.

The paper is organised as follows. In § 2 the model is defined and in § 3 the susceptibility series are discussed. Section 4 contains an analysis of these series yielding determinations of the critical exponent, critical point and critical amplitude. In § 5 we discuss the high-temperature series for the free energy and specific heat. Section 6 contains our conclusions.

2. The model

For our reference model, we shall take the (classical) O(2) spin model (also called the plane rotator model), with Hamiltonian

$$\mathscr{H} = -J \sum_{\langle ij \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j - \boldsymbol{H} \cdot \sum_i \boldsymbol{S}_i \tag{2.1}$$

where $\langle ij \rangle$ denotes nearest-neighbour pairs of lattice sites i and j. To study the effect of a discrete interaction, a simple and natural choice is to replace the usual scalar product spin-spin interaction by $\mathrm{sgn}(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)$. This yields a model defined by the partition function

$$Z = \int \prod_i \mathrm{d}\Omega_i^{(1)} \, \mathrm{e}^{-\beta \mathscr{H}} \tag{2.2a}$$

where $\int \mathrm{d}\Omega^{(1)} \equiv (2\pi)^{-1} \int_{-\pi}^{\pi} \mathrm{d}\theta$ is the unit-normalised measure on S^1, $\beta \equiv (k_{\mathrm{B}} T)^{-1}$ and

$$\mathscr{H} = -J \sum_{\langle ij \rangle} \mathrm{sgn}(\boldsymbol{S}_i \cdot \boldsymbol{S}_j) - \boldsymbol{H} \cdot \sum_i \boldsymbol{S}_i . \tag{2.2b}$$

Evidently, the model (2.2) shares features of both the usual O(2) and Ising models: although the symmetry group of the zero-field Hamiltonian is O(2) and the spins $\boldsymbol{S}_i \in S^1$, the spin-spin interaction takes on the same values as in the Ising model. We shall see how this twofold nature will manifest itself in the properties of the model. We first observe that the spin-spin interaction has a finite two-term character expansion, just as in the regular Ising model, and in contrast to the infinite-series character expansion (in terms of modified Bessel functions) for the regular O(2) model. Hence, we can rewrite the partition function (2.2) as

$$Z = (\cosh K)^{N_l} \int \prod_k \mathrm{d}\Omega_k^{(1)} \exp\left(\beta \boldsymbol{H} \cdot \sum_n \boldsymbol{S}_n \right) \prod_{\langle ij \rangle} [1 + v \, \mathrm{sgn}(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)] \tag{2.3a}$$

where

$$v \equiv \tanh K \tag{2.3b}$$

with $K \equiv \beta J$, and where N_l denotes the number of links in the lattice.

For comparison, the model studied by Guttmann and Joyce (1973) and Guttmann and Nymeyer (1978) is defined by the partition function (2.2a) with

$$\mathscr{H} = -J \sum_{\langle ij \rangle} \mathrm{sgn}(\boldsymbol{S}_i \cdot \boldsymbol{S}_j) - \sum_i \mathrm{sgn}(\boldsymbol{H} \cdot \boldsymbol{S}_i) \tag{2.4}$$

i.e. the standard coupling to an external field is also replaced by a discretised form. One may ask whether this discretisation of the coupling to an external field changes the universality class of the model (2.4) relative to that of (2.2). As discussed below, our study answers this question.

3. High-temperature susceptibility series

In order to investigate the critical properties of the model (2.2), we have calculated high-temperature series expansions for the (zero-field, isothermal) susceptibility χ on the SC, BCC and FC lattices. As usual, it is convenient to define the reduced susceptibility

$$\bar{\chi} \equiv \beta^{-1} \chi \tag{3.1}$$

whence

$$\bar{\chi} = 1 + 2 \sum_{(i,j);\, i \neq j} \langle \boldsymbol{S}_i \cdot \boldsymbol{S}_j \rangle \tag{3.2}$$

(where each pair $(i, j) = (j, i)$ is counted only once in the sum). Let us for the moment consider the $O(N)$ generalisation of (2.2)-(3.2). The graphs contributing to (3.2) may be classified as chain and non-chain graphs. For the former, we have found the important theorem that (for arbitrary dimensionality and lattice type) the corresponding integrals

$$I(c_l) = \int \prod_{k=1}^{l+1} \mathrm{d}\Omega_k^{(1)} \boldsymbol{S}_1 \cdot \boldsymbol{S}_{l+1} \prod_{\langle m,n \rangle \in c_l} \operatorname{sgn}(\boldsymbol{S}_m \cdot \boldsymbol{S}_n)$$
$$= R_N^l \tag{3.3a}$$

where $\langle m, n \rangle$ denote successive nearest-neighbour links in the chain graph c_l and

$$R_N = \pi^{-1/2} \frac{\Gamma(\frac{1}{2}N)}{\Gamma(\frac{1}{2}(N+1))}. \tag{3.3b}$$

We have proved this by an iterative argument, starting with the one-link case. It follows that a natural variable for the high-temperature series expansion for the susceptibility of the $O(N)$ generalisation of (2.2) is

$$v_N \equiv R_N v. \tag{3.4}$$

Furthermore, writing

$$\bar{\chi}_\Lambda = 1 + \sum_{l=1}^{\infty} b_{\Lambda,l} v_N^l \tag{3.5}$$

(where the subscript Λ denotes the lattice type) it follows, as a first corollary of theorem (3.3), that for the $O(N)$ generalisation of (2.2)

$$b_{\Lambda,l} = (b_{\Lambda,l})_{\text{Ising}} \qquad \text{for } l < l_{\Lambda,p} \tag{3.6}$$

where l_p denotes the first order where a non-chain graph contributes to χ. These are

$$l_{\Lambda,p} = \begin{cases} 4 \\ 5 \end{cases} \qquad \text{for } \Lambda = \begin{cases} \text{FCC} \\ \text{BCC or SC.} \end{cases} \tag{3.7}$$

As a second corollary,

$$\chi(v_N) = \chi_{\text{Ising}}(v \to v_N) \qquad \text{for } \Lambda = \text{Cayley tree} \tag{3.8}$$

i.e. the susceptibility is given by the Ising susceptibility with the replacement of v by v_N. In particular, for the case of a $d = 1$ lattice, we obtained the exact result for the $O(N)$ generalisation of (2.2) that

$$\bar{\chi} = \frac{1 + v_N}{1 - v_N}. \tag{3.9}$$

Since $R_N < 1$ for all $N \geqslant 2$, it follows from (3.9) that for this case $\bar{\chi}$ is never singular, in sharp contrast to the $d = 1$ Ising and $O(N \geqslant 2)$ models, where $\bar{\chi}$ has, respectively, essential and algebraic divergences at $T = 0$. The two-spin correlation function was calculated to be

$$\langle \boldsymbol{S}_0 \cdot \boldsymbol{S}_r \rangle = v_N^r \tag{3.10}$$

so that the correlation length is

$$\xi = -1/\ln v_N \tag{3.11}$$

which never diverges, even at $T=0$, where it reaches its maximal value:

$$\xi(T=0) = -1/\ln R_N. \tag{3.12}$$

These findings generalised the $N=2$ results of Guttmann and Joyce (1973).

Returning to the O(2) case under study here, we use the expansion variable

$$\bar{v} \equiv v_2 = 2v/\pi \tag{3.13}$$

and write the reduced susceptibility as

$$\bar{\chi}_\Lambda = 1 + \sum_{l=1}^{\infty} b_{\Lambda,l}\bar{v}^l. \tag{3.14}$$

We have calculated the series coefficients $b_{\Lambda,l}$ to order $l=7$ for a general lattice and $l=8$ for the simple cubic (square and hypercubic) lattice. The results were given for the three $d=3$ lattices, FCC, BCC and SC, in Lee and Shrock (1987). For the reader's convenience, they are also included here in tables 1-3. The $d=2$ results will be analysed in a separate paper (Lee and Shrock 1988).

Table 1. Susceptibility coefficients $b_{\mathrm{FCC},l}$ for the FCC lattice.

l	$b_{\mathrm{FCC},l}$
1	12
2	132
3	1404
4	$14\,700-3\pi^3$
5	$152\,532-36\pi^2-54\pi^3-7\pi^4$
6	$1573\,716-180\pi^2-1071\pi^3-\frac{251}{2}\pi^4-\frac{255}{16}\pi^5$
7	$16\,172\,148-6954\pi^2-12\,432\pi^3-2607\pi^4-\frac{4425}{16}\pi^5-\frac{369}{10}\pi^6$

Table 2. Susceptibility coefficients $b_{\mathrm{BCC},l}$ for the BCC lattice.

l	$b_{\mathrm{BCC},l}$
1	8
2	56
3	392
4	2648
5	$17\,960-2\pi^4$
6	$120\,056-36\pi^2-23\pi^4$
7	$804\,824+216\pi^2-300\pi^4-\frac{9}{2}\pi^6$

Table 3. Susceptibility coefficients $b_{\mathrm{SC},l}$ for the SC lattice.

l	$b_{\mathrm{SC},l}$
1	6
2	30
3	150
4	726
5	$3534-\frac{1}{2}\pi^4$
6	$16\,926-\frac{9}{2}\pi^4$
7	$81\,390+54\pi^2-\frac{77}{2}\pi^4-\frac{5}{8}\pi^6$
8	$387\,966+84\pi^2-222\pi^4-\frac{51}{10}\pi^6$

Because of the property (3.6), the susceptibility series have the interesting and unusual feature that in low orders they are Ising-like (in terms of the variable v_N) whereas, as l increases above $l_{\Lambda,p}$, they deviate from the corresponding Ising series and exhibit the true thermodynamic properties of the model (2.2).

4. Analysis of series

4.1. General

We have analysed the series (3.11) for the FCC, BCC and SC lattices using the ratio test, Neville tables and Padé approximants. These methods are reviewed in Gaunt and Guttmann (1974) and Baker (1975). The critical singularities are assumed to be of the generic form

$$\bar{\chi}(\bar{v})_{\mathrm{FCC,sing}} \sim A(\bar{v})_{\mathrm{FCC}}[1-\bar{v}/(\bar{v}_c)_{\mathrm{FCC}}]^{-\gamma} \tag{4.1a}$$

for the close-packed FCC lattice, and of the form

$$\bar{\chi}(\bar{v})_{\Lambda,\mathrm{sing}} \sim A(\bar{v})_{\Lambda}[1-\bar{v}/(\bar{v}_c)_{\Lambda}]^{-\gamma} + B(\bar{v})_{\Lambda}[1+\bar{v}/(\bar{v}_c)_{\Lambda}]^{\theta} \tag{4.1b}$$

for the loose-packed BCC and SC lattices, where Λ denotes lattice type, and $A(\bar{v})_\Lambda$ and $B(\bar{v})_\Lambda$ are analytic. The lattice independence of the critical exponent γ, as indicated in (4.1), will be shown to be in agreement with our results. The second term in (4.1*b*) reflects the presence of the antiferromagnetic critical point at $v=-v_c$, where v_c is the ferromagnetic critical point. Although general arguments analogous to those given for the Ising model (Danielian 1964) imply that there is an antiferromagnetic phase transition for the model (2.2) on the FCC lattice, our series, as will be seen, do not give evidence for a critical singularity associated with a second-order phase transition at this point. This is similar to what was found for the $d=3$ Ising model (Sykes *et al* 1972) and suggests that, for the model (2.2), the antiferromagnetic transition on a FCC lattice is first order, just as it has been shown to be (Phani *et al* 1980, Polgreen 1984) for the Ising model on the FCC lattice.

Using the expansion

$$A(\bar{v})_\Lambda = \sum_{n=0}^{\infty} A_{\Lambda,n}(\bar{v}-\bar{v}_c)^n \tag{4.2a}$$

and defining the critical amplitude as

$$A_\Lambda \equiv A_{\Lambda,0} \tag{4.2b}$$

and similarly for $B(\bar{v})$, the generic form for the dominant singularity of the susceptibility can be expressed as

$$\bar{\chi}(\bar{v})_\Lambda \sim A_\Lambda[1-\bar{v}/(\bar{v}_c)_\Lambda]^{-\gamma}. \tag{4.3}$$

The higher orders of the Padé tables for $\mathrm{d}\ln\bar{\chi}(\bar{v})/\mathrm{d}\bar{v}$ are given in tables 4–6 for the FCC, BCC and SC lattices, respectively. In the $[\mathcal{N},\mathcal{D}]$ entry in each table the upper number is the pole at $\bar{v}=\bar{v}_c$ and the lower number is minus the residue at this pole, namely the exponent γ. We recall that, for a given order, the diagonal and near-diagonal entries in a Padé table are given the most weight because the diagonal $[\mathcal{N},\mathcal{N}]$ entry is invariant under Euler transformations of the form $\bar{v}\to\bar{v}'=a\bar{v}/(b\bar{v}+c)$, which are often used to improve results obtained from analysis of series.

Table 4. Padé table for $d\ln\bar{\chi}(\bar{v})/d\bar{v}$ for the FCC lattice. In each $[\mathcal{N}, \mathcal{D}]$ entry, the upper number is the pole at $\bar{v} = \bar{v}_c$ and the lower is the corresponding value of γ. The symbol 'a' indicates an approximant with a spurious nearly coincident pole-zero pair near to the origin.

	$\mathcal{N}$					
$\mathcal{D}$	0	1	2	3	4	5
1				0.103 725 1.329 3	0.103 795 1.333 8	0.103 774 1.332 2
2			0.103 760 1.331 6	0.103 819 1.335 7	0.103 779 1.332 6	
3		0.103 835 1.336 7	0.103 810 1.335 0	0.103 787 1.333 3		
4	0.103 822 1.335 8	0.103 813 1.335 2	0.103 872[a] 1.338 1[a]			
5	0.103 812 1.335 2	0.103 825[a] 1.336 0[a]				
6	0.103 773 1.332 1					

Table 5. Padé table for $d\ln\bar{\chi}(\bar{v})/d\bar{v}$ for the BCC lattice. In each $[\mathcal{N}, \mathcal{D}]$ entry, the upper number is the pole at $\bar{v} = \bar{v}_c$ and the lower is the corresponding value of γ.

	$\mathcal{N}$					
$\mathcal{D}$	0	1	2	3	4	5
1				0.150 068 1.022 5	0.167 253 1.758 2	0.152 880 1.025 5
2			0.157 875 1.292 0	0.159 178 1.336 9	0.159 252 1.340 0	
3		0.158 464 1.313 2	0.160 034 1.378 4	0.159 255 1.340 2		
4	0.153 792 1.144 3	0.159 565 1.354 7	0.159 348 1.345 0			
5	0.165 326 1.702 7	0.159 306 1.342 8				
6	0.156 119 1.178 3					

4.2. *Critical exponent* γ

As usual, the results for the critical point, exponent and amplitude extracted from the FCC series show the most rapid approach to fixed values. As can be seen from table 4, by the order $l = 7$ in (3.5), the FCC series already yields quite stable values for γ and $\bar{v}_c$. Using tables 4–6, together with Neville tables which we have calculated, we infer the susceptibility critical exponents $\gamma = 1.33 \pm 0.01$ for FCC, 1.34 ± 0.04 for BCC and 1.37 ± 0.07 for SC. Since the FCC series gives by far the most stable results, our overall determination of γ is dominated by the FCC value. We find

$$\gamma = 1.33 \pm 0.01 \tag{4.4}$$

Table 6. Padé table for $d \ln \bar{\chi}(\bar{v})/d\bar{v}$ for the SC lattice. In each $[\mathcal{N}, \mathcal{D}]$ entry, the upper number is the pole at $\bar{v} = \bar{v}_c$ and the lower is the corresponding value of γ. The symbol 'cp' denotes a complex pair of poles. (For example, in the [2, 3] entry, the pair is $0.2442 \pm 0.0160i$.)

$\mathcal{D}$ \ $\mathcal{N}$	0	1	2	3	4	5	6
1				0.209 318 1.013 6	0.235 409 1.823 6	0.215 221 1.064 9	0.230 530 1.722 7
2			0.219 853 1.272 6	0.222 719 1.343 0	0.223 776 1.375 8	0.223 780 1.376 0	
3		0.220 371 1.285 9	cp —	0.224 216 1.393 7	0.223 780 1.376 0		
4	0.216 854 1.193 0	0.223 209 1.360 7	0.224 532 1.407 6	0.223 578 1.366 8			
5	0.231 093 1.694 7	0.224 258 1.395 9	0.223 995 1.385 6				
6	0.220 839 1.259 7	0.223 888 1.380 8					
7	0.225 889 1.485 4						

which is lattice independent, to within the uncertainties cited for all of the three lattice types. For comparison, high-temperature series expansions for the regular 3D classical O(2) (plane rotator) model give $\gamma = 1.318 \pm 0.010$ (Ferer *et al* 1973, see also Bowers and Joyce 1967), while a combined analysis of high-temperature series expansions for the 3D plane rotator model and the 3D $S = \infty$ quantum XY model, which should be in the same $d = 3$, $N = 2$ universality class, yields $\gamma = 1.333 \pm 0.010$ (Rogiers *et al* 1978, 1979). A field-theoretic method using the ε expansion gives, for the $d = 3$, $N = 2$ universality class, the values $\gamma = 1.316 \pm 0.0025$ and $\gamma \simeq 1.324$, depending on the method (Le Guillou and Zinn-Justin 1980). We conclude that the value (4.4) is equal, to within the uncertainty, to the susceptibility exponent for the regular 3D O(2) model. Thus, to the accuracy with which we have determined the universality class of the model (2.2), it is the same as that of the regular 3D O(2) model. In passing, it may also be noted that the value (4.4) is strongly inconsistent with the value of γ for the 3D O(N) model with any other value of N than $N = 2$.

It is also of interest to compare our result with that found previously for the model (2.4). The closest quantity to a usual reduced susceptibility in model (2.4) is

$$1 + 2 \sum_{(i,j);\, i \neq j} \langle \mathrm{sgn}(\boldsymbol{S}_i \cdot \hat{\boldsymbol{H}})\, \mathrm{sgn}(\boldsymbol{S}_j \cdot \hat{\boldsymbol{H}}) \rangle. \tag{4.5}$$

Instead, Guttmann and Joyce (1973) and Guttmann and Nymeyer (1978) studied the function

$$1 + 2 \sum_{(i,j);\, i \neq j} \langle \mathrm{sgn}(\boldsymbol{S}_i \cdot \boldsymbol{S}_j) \rangle \tag{4.6}$$

since the series for the latter function is easier to calculate. As compared with our susceptibility series, the series for (4.6) is much simpler; for example, the series coefficients are integers, as in the Ising model. In contrast, our susceptibility series coefficients for the model (2.2) are polynomials in π, the complexity of which increases with increasing order. When the function (4.6) was fit to a singularity of the form

$(1-v/v_c)^{-\lambda}$, it was found that $\lambda = 1.335 \pm 0.010$, consistent with being lattice independent. Moreover it was conjectured by Guttmann and Joyce (1973) that the singularity of the function (4.6) would be the same as that of (4.5). Comparing this result with our determination (4.4), we find that the critical exponent defined by the singularity in (4.6) is equal, to within the uncertainty, to the susceptibility exponent γ for the model (2.2). Accepting the suggestion that (4.6) and (4.5) have the same singularities, this indicates that models (2.2) and (2.4) are in the same universality class (the zero-field properties are, of course, identical).

4.3. Critical point

Our Padé and Neville table analyses of the high-temperature susceptibility series yield the following values for the ferromagnetic critical points of the three lattices:

$$(\bar{v}_c)_{\mathrm{FCC}} = 0.1038 \pm 0.0002 \tag{4.7a}$$

$$(\bar{v}_c)_{\mathrm{BCC}} = 0.159 \pm 0.001 \tag{4.7b}$$

$$(\bar{v}_c)_{\mathrm{SC}} = 0.224 \pm 0.002 \tag{4.7c}$$

and the resultant values for K_c listed in table 7. These results are in excellent agreement with the values obtained for the FCC lattice by Guttmann *et al* (1972) and Guttmann and Joyce (1973), and for the BCC and SC lattices by Guttmann and Nymeyer (1978), using the high-temperature series for the function (4.6).

Table 7. Inverse critical temperature $(K_c)_\Lambda$ for sgn O(2) model on FCC, BCC and SC lattices. For comparison, values of $(K_c)_\Lambda$ for the $d=3$ Ising and regular O(2) (plane rotator) models are listed; these are from (a) (Sykes *et al* 1972) and (b) (Ferer *et al* 1973), respectively. The latter have typical uncertainties of 1 in the last digit.

Lattice Λ	Ising[a]	sgn O(2), this work	O(2)[b]
FCC	0.161 19	0.1645 ± 0.0002	0.2075
BCC	0.250 33	0.255 ± 0.002	0.320
SC	0.357 1	0.368 ± 0.004	0.454

Table 7 shows a comparison of the values for the critical point of the sgn O(2) model determined from our study with the corresponding critical point values for the 3D Ising and regular O(2) models. It indicates that

$$(K_c)_{\Lambda,\mathrm{Ising}} < (K_c)_{\Lambda,\mathrm{sgnO(2)}} < (K_c)_{\Lambda,\mathrm{O(2)}} \tag{4.8}$$

for each lattice type Λ. We can explain this as follows. It should first be noted that the short-range order in the sgn O(N) model actually builds up more rapidly, as the temperature decreases from infinity, than in the regular O(N) model. This is demonstrated by the lowest-order high-temperature expansions of the nearest-neighbour correlation functions in these respective models: for the general O(N) case and an arbitrary lattice type,

$$\langle \boldsymbol{S}_i \cdot \boldsymbol{S}_{i_{nn}} \rangle = K/N + \mathrm{O}(K^3) \qquad \text{for the regular O}(N)\text{ model} \tag{4.9}$$

where i and i_{nn} denote a nearest-neighbour pair of sites, whereas

$$\langle \boldsymbol{S}_i \cdot \boldsymbol{S}_{i_{nn}} \rangle = R_N K + \mathrm{O}(K^3) \qquad \text{for the sgn O}(N)\text{ model} \tag{4.10}$$

where R_N was given in (3.3*b*). Now, since

$$R_N > 1/N \qquad \forall N \geq 2 \tag{4.11}$$

it follows, as stated, that for general $N \geq 2$, the short-range order (which is measured, for example, by the nearest-neighbour two-spin correlation function) grows more rapidly with K in the sgn O(N) model (2.2) than in the usual O(N) model (2.1). (Here and elsewhere, it is assumed that $N \geq 2$, since the O(1) model is, of course, just the Ising model and does not have continuous spin variables.) This behaviour is, at first sight, rather remarkable, since the spin-spin interaction in the sgn O(N) model allows a far more disordered spin configuration than that of the regular O(N) model. However, the result (4.10) can be understood as due to the fact that the low orders in the high-temperature series expansion of $\langle S_i \cdot S_{i_{nn}} \rangle$ are Ising like in the scaled variable v_N. Evidently, the more rapid build-up of the short-range order in the model (2.2), as compared with the regular O(2) model, leads to the phase transition and onset of long-range order at a higher temperature than in the latter model. The first part of the inequality (4.8) is also interesting. One would clearly expect that, since there is more intrinsic disorder in the sgn O(2) model than the Ising model, the critical temperature for the former should be lower than that for the latter model, and (4.8) confirms this expectation. What is interesting is how close the critical points are for these two models; for each of the three lattice types Λ, $(K_c)_{\Lambda,\mathrm{sgnO(2)}}$ is only about 2% larger than the respective value $(K_c)_{\Lambda,\mathrm{Ising}}$.

The greater intrinsic disorder in the sgn O(N) model with respect to the regular O(N) model manifests itself in the property that, at zero temperature, neither the short-range order nor the long-range order saturate at unity. Instead, as we have shown by Monte Carlo simulations,

$$\langle S_i \cdot S_{i_{nn}} \rangle (T=0) = 0.75 \pm 0.01 \tag{4.12}$$

and

$$M(T=0) = 0.83 \pm 0.02. \tag{4.13}$$

4.4. Critical amplitudes

We have calculated the high-temperature series for $\bar{\chi}(\bar{v})_\Lambda^{1/\gamma}$ in order to obtain the critical amplitudes. Since these series should have simple pole singularities, it is not necessary to take a logarithmic derivative before calculating the Padé approximants. Hence, one gains in sensitivity, although at the price of biasing the result via the input value of γ. In tables 8–10, we give the Padé tables for the FCC, BCC and SC series for $\bar{\chi}(\bar{v})_\Lambda^{1/\gamma}$, using the central value $\gamma = 1.33$ in (4.4). We have also calculated corresponding series and Padé tables using the upper and lower 1σ limits in (4.4). From a comparative analysis of all of these, we obtain the results

$$(\bar{v}_c)_{\mathrm{FCC}} A_{\mathrm{FCC}}^{1/1.33} = 0.0959 \pm 0.0010 \tag{4.14a}$$

$$(\bar{v}_c)_{\mathrm{BCC}} A_{\mathrm{BCC}}^{1/1.33} = 0.1471 \pm 0.0020 \tag{4.14b}$$

$$(\bar{v}_c)_{\mathrm{SC}} A_{\mathrm{SC}}^{1/1.33} = 0.220 \pm 0.010. \tag{4.14c}$$

Substituting the respective values of $\bar{v}_c$ from the $\mathrm{d}\ln\bar{\chi}(\bar{v})/\mathrm{d}\bar{v}$ Padé analysis, (4.7) or equivalently, to within the quoted uncertainties, directly from the poles of $\bar{\chi}(\bar{v})^{1/\gamma}$, and solving for A_Λ, we calculate the critical amplitudes for the susceptibility listed in table 11. We find that the critical amplitudes for the FCC and BCC lattices are consistent

Table 8. Padé table for $\bar{\chi}(\bar{v})_{\mathrm{FCC}}^{1/1.33}$. In each $[\mathcal{N}, \mathcal{D}]$ entry, the upper number is the pole at $\bar{v} = \bar{v}_c$ and the lower one is minus the residue at this pole. The symbol 'a' is defined in table 4.

	$\mathcal{N}$						
$\mathcal{D}$	0	1	2	3	4	5	6
1				0.103 723 0.095 744	0.103 735 0.095 798	0.103 746 0.095 862	0.103 750 0.095 886
2			0.103 723 0.095 743	0.103 731 0.095 776	cp —	0.103 752 0.095 900	
3		0.103 736 0.095 802	0.103 736 0.095 802	0.103 908 0.099 052	0.103 748 0.095 867		
4	0.103 757 0.095 901	0.103 736 0.095 802	0.103 736[a] 0.095 802[a]	0.103 752 0.095 899			
5	0.103 738 0.095 816	0.103 743 0.095 842	0.103 750 0.095 885				
6	0.103 745 0.095 853	0.103 756 0.095 933					
7	0.103 749 0.095 879						

Table 9. Padé table for $\bar{\chi}(\bar{v})_{\mathrm{BCC}}^{1/1.33}$. In each $[\mathcal{N}, \mathcal{D}]$ entry, the upper number is the pole at $\bar{v} = \bar{v}_c$ and the lower one is minus the residue at this pole. The symbol 'a' is defined in table 4.

	$\mathcal{N}$						
$\mathcal{D}$	0	1	2	3	4	5	6
1				0.160 288 0.152 50	0.158 138 0.142 54	0.159 660 0.150 98	0.158 662 0.144 49
2			0.159 195 0.147 65	0.158 901 0.146 47	0.159 027 0.147 08	0.159 057 0.147 25	
3		0.159 474 0.148 79	0.158 851 0.146 22	0.158 995 0.146 90	0.159 064 0.147 30		
4	0.160 253 0.152 16	0.159 025 0.147 06	0.159 029 0.147 08	0.159 050 0.147 20			
5	0.158 214 0.143 12	0.159 029 0.147 08	0.159 024[a] 0.147 06[a]				
6	0.159 638 0.150 75	0.159 048 0.147 19					
7	0.158 670 0.144 64						

with being equal, to within their uncertainties, and are slighly smaller than the critical amplitude for the sc lattice. This is the same behaviour as was found for the critical susceptibility amplitudes in the 3D Ising and regular O(2) (plane rotator) model; for comparison, these critical amplitudes are also listed in table 11. Moreover, we observe the general inequality

$$(A_{\Lambda})_{\mathrm{Ising}} > (A_{\Lambda})_{\mathrm{sgn\,O(2)}} > (A_{\Lambda})_{\mathrm{O(2)}}. \tag{4.15}$$

It is interesting that, as was the case with the critical coupling K_c, the values of the critical amplitude for the sgn O(2) model and Ising model are rather close. For

Table 10. Padé table for $\bar{\chi}(\bar{v})_{SC}^{1/1.33}$. In each $[\mathcal{N}, \mathcal{D}]$ entry, the upper number is the pole at $\bar{v} = \bar{v}_c$ and the lower one is minus the residue at this pole.

$\mathcal{D}$	$\mathcal{N}$: 1	2	3	4	5	6
2			0.222 061	0.222 284	0.222 573	0.222 725
			0.211 96	0.213 07	0.214 80	0.215 86
3		0.221 770	0.222 245	cp	0.223 172	
		0.210 40	0.212 85	—	0.220 49	
4	0.222 254	0.222 316	0.222 562	0.223 118		
	0.212 95	0.213 25	0.214 68	0.219 80		
5	0.222 320	0.222 218[a]	0.223 718			
	0.213 27	0.212 83[a]	0.229 56			
6	0.222 532	0.225 626				
	0.214 51	0.304 71				
7	0.222 694					
	0.215 614					

Table 11. Critical amplitudes A_Λ for the susceptibility of the sgn O(2) model (2.2) on FCC, BCC and SC lattices. For comparison, critical amplitudes for the susceptibilities of the $d = 3$ Ising and usual O(2) (plane rotator) models are listed; values are from (a) Sykes *et al* (1972) and (b) Ferer *et al* (1973), respectively.

Lattice Λ	Ising[a]	sgn O(2)	O(2)[b]
FCC	0.963 ± 0.002	0.900 ± 0.010	0.4577 ± 0.0010
BCC	0.967 ± 0.003	0.902 ± 0.015	0.466 ± 0.002
SC	1.1016 ± 0.0010	0.976 ± 0.050	0.5135 ± 0.0025

reference, the critical amplitudes for the function (4.6) were calculated for the BCC and SC lattices by Guttmann and Nymeyer (1978) and are $A_{(4.6),\mathrm{BCC}} = 1.081 \pm 0.004$ and $A_{(4.6),\mathrm{SC}} = 1.121 \pm 0.003$.

4.5. *Antiferromagnetic singularities*

It is also of interest to investigate the singularities in the antiferromagnetic region ($J < 0$). Since the free energy has a singularity, for loose-packed lattices, at $v = -v_c$, the susceptibility will also. However, it is not expected that this singularity at $v = -v_c$ in the (uniform) susceptibility will be divergent. This is obvious from physical considerations: at $v = -v_c$, the system is on the verge of antiferromagnetic ordering, so there should not be any particularly large response to a uniform applied field. (Of course, the staggered susceptibility obeys the symmetry relation $\chi(-v)_{\mathrm{stagg}} = \chi(v)$ and thus is divergent at this point.) For the close-packed FCC lattice, as discussed above, although there is an antiferromagnetic phase transition, series expansions for χ do not give evidence for it, indicating that it is first order. We thus concentrate here on the loose-packed BCC and SC lattices. In table 12 we show relevant results from Padé approximants for $\mathrm{d} \ln \chi(\bar{v})/\mathrm{d}\bar{v}$ on the SC lattice. For each entry, the uppermost listing is the pole in $\bar{v}$ at the ferromagnetic singularity. It is well known (see, for example, Gaunt and Guttmann 1974) that Padé approximants for $\mathrm{d} \ln \bar{\chi}(\bar{v})/\mathrm{d}\bar{v}$ will try to fit a

Table 12. Padé table showing the antiferromagnetic singularity in $d\bar{\chi}(\bar{v})_{SC}/d\bar{v}$. In each entry $[\mathcal{N}, \mathcal{D}]$, the uppermost listing is the pole representing the ferromagnetic singularity, the middle listing is the pole near the antiferromagnetic critical point and the lowermost listing is the zero at a slightly larger value of $-\bar{v}$.

$\mathcal{D}$	$\mathcal{N}$ = 1	2	3	4	5
2			0.222 72	0.223 78	0.223 78
			−0.248 62	−0.292 61	−0.292 36
			−0.271 72	−0.373 09	−0.371 90
3		cp	0.224 22	0.223 78	
		−0.3083	−0.280 03	−0.292 37	
		−0.4035	−0.329 85	−0.371 91	
4	0.223 21	0.224 53	0.223 58		
	−0.235 92	−0.274 86	−0.286 77		
	−0.250 54	−0.317 01	−0.347 49		
5	0.224 26	0.224 00			
	−0.282 98	−0.309 38			
	−0.339 46	cp			
6	0.223 89				
	−0.297 28				
	−0.401 20				

finite branch-point singularity in $\bar{\chi}(\bar{v})$ of the form in (4.1*b*), $[1+\bar{v}/(\bar{v}_c)_\Lambda]^\theta$, and its associated branch cut, by a pole near $\bar{v}=-(\bar{v}_c)_\Lambda$ followed by a sequence of alternating zeros and poles outward toward larger $-\bar{v}$. In table 12, we give as the middle listing the pole near the antiferromagnetic singularity and, as the lowermost listing, the zero at a slightly larger value of $-\bar{v}$. These show clearly the expected behaviour for a finite branch-point singularity in $\bar{\chi}(\bar{v})$. Note that, in order for a Padé approximant to $d\ln\bar{\chi}(\bar{v})/d\bar{v}$ to be able to give an adequate description of the branch-point singularity at $\bar{v}=-\bar{v}_c$, it must have at least two poles (one for the ferromagnetic pole and one for at least the first pole in the pole-zero sequence) and at least one zero. That is, the $[\mathcal{N}, \mathcal{D}]$ Padé approximant must have $\mathcal{N}\geq 1$ and $\mathcal{D}\geq 2$. Only such entries are included in table 12. We have found similar results for the BCC susceptibility series.

5. High-temperature series expansions for the free energy and specific heat

Definining the reduced free energy as

$$f \equiv \lim_{N_s\to\infty}\frac{1}{N_s}\ln Z \tag{5.1}$$

we can write the high-temperature series expansion for f (with zero external field) in the convenient form

$$f_\Lambda = \tfrac{1}{2}q_\Lambda \ln(\cosh K) + \sum_{l=3}^{\infty} a_{\Lambda,l}v^l \tag{5.2}$$

for each of the three lattice types Λ = FCC, BCC and SC, where q_Λ denotes the coordination number of the respective lattices. With the expansion variable v, as in (5.2), the coefficients a_l are rational. The high-temperature series for the free energy is thus qualitatively simpler than that for the susceptibility, which involves polynomials in π (and is Ising like in low orders if the expansion variable is taken as $\bar{v}$). The coefficients a_l were calculated to order v^9 for a general lattice and v^{10} for loose-packed lattices and were listed for FCC, BCC and SC lattices in Lee and Shrock (1987). To render the present discussion self-contained, we include these results here as table 13. We recall that, because of the well known mapping between ferromagnetic and antiferromagnetic models on loose-packed lattices, it follows for these that $f(-v) = f(v)$, whence $a_{\mathrm{odd}\, l} = 0$. From f, one computes the specific heat per site, C, in the usual way. We write (with $k_B = 1$)

$$C_\Lambda = \sum_{l=2}^{\infty} c_{\Lambda,l} v^l. \tag{5.3}$$

Our results for the $c_{\Lambda,l}$ are listed in table 14. Although the magnetic-field-dependent properties of our sgn O(2) model and the model (2.4) are different, the zero-field properties are the same. Accordingly, we have compared our zero-field free energy series with the calculations by Guttmann and Joyce (1973, see their table 1) for the FCC lattice and Guttmann and Nymeyer (1978, see their table 1) for the BCC and SC lattices. We have been informed (Guttmann 1987 private communication) that the FCC coefficients listed for the O(2) step (sgn) model in table 1 of Guttmann and Joyce (1973) actually referrred to the quantity $(v/K)^2 C/k_B$ rather than C/k_B. (The coefficients for the Ising and regular O(2) models, however, did refer to C/k_B.) Transforming the series to account for this factor $(v/K)^2$, we find exact agreement to

Table 13. Coefficients $a_{\Lambda,l}$ for the free energy, as defined in (5.2).

l	SC	BCC	FCC
3	0	0	4
4	1	4	11
5	0	0	35
6	$\frac{44}{15}$	$\frac{296}{15}$	$\frac{358}{3}$
7	0	0	$\frac{13217}{30}$
8	$\frac{1891}{210}$	$\frac{4244}{35}$	$\frac{122567}{70}$
9	0	0	$\frac{925343}{126}$
10	$\frac{10616}{315}$	$\frac{125408}{135}$	—

Table 14. Specific heat coefficients $c_{\Lambda,l}$ defined by equation (5.3).

l	SC	BCC	FCC
2	3	4	6
3	0	0	24
4	11	$\frac{140}{3}$	130
5	0	0	644
6	$\frac{953}{15}$	$\frac{22292}{45}$	$\frac{49726}{15}$
7	0	0	$\frac{258491}{15}$
8	$\frac{12469}{35}$	$\frac{1821676}{315}$	$\frac{3216518}{35}$
9	0	0	$\frac{156924851}{315}$
10	$\frac{1179103}{525}$	$\frac{114665212}{1575}$	—

$O(v^9)$. (These authors also list an approximate numerical result for the FCC $O(v^{10})$ term.) As far as our SC specific heat series extends, it also agrees with that given (numerically, to $O(v^{14})$ with an approximate numerical result for $O(v^{16})$) by Guttmann and Nymeyer (1978). Our BCC specific heat series disagrees past $O(v^2)$ with the series given by Guttmann and Nymeyer (1978). We are informed (Guttmann 1987 private communication) that the BCC series in Guttmann and Nymeyer (1978) is in error, and the corrected series agrees with ours to $O(v^{10})$ (approximate numerical values were also given for the $O(v^{12})$ and $O(v^{14})$ BCC terms).

As usual with specific heat series, these do not yield very precise determinations of the critical exponent α, but we can infer that α is consistent with the values found for the regular 3D O(2) model (namely $\alpha = -0.02 \pm 0.03$ from high-temperature series expansions (Ferer *et al* 1973), $\alpha = -0.007 \pm 0.006$ from field-theoretic methods (Le Guillou and Zinn-Justin 1980)).

6. Conclusions

In summary, we have presented the details of an analysis of the high-temperature series for the susceptibility of the 3D sgn O(2) model. This work indicates that, to within the accuracy with which we have determined it, the susceptibility exponent γ for this model is the same as that for the regular 3D O(2) model. Our results also show that the 3D sgn O(2) model with the standard coupling to an external field is in the same universality class as a previously studied model (Guttmann and Joyce 1973, Guttmann and Nymeyer 1978) which has not only a discretely valued spin-spin interaction, but also a discretely valued coupling to an external field. In conjunction with an analysis of the specific heat, these results indicate that, for $d = 3$, the O(2) model with discretely valued spin-spin interactions is in the same universality class as the regular $d = 3$ O(2) model. We have also analysed the susceptibility series for the sgn O(2) model on a $d = 4$ hypercubic lattice and have again found that γ is consistent with being equal to unity, thereby showing that (i) for this case also the sgn O(2) model is in the same universality class as the regular O(2) model; and (ii) the upper critical dimensionality of the sgn O(2) model is $d = 4$, as for regular spin models.

These results for the 3D O(2) model may be contrasted with two exact results. First, consider the Gaussian model. For an arbitrary dimensionality d and lattice type Λ, if one switches from the usual continuous interaction $\varepsilon_i \varepsilon_j$ to the discretely valued interaction $\mathrm{sgn}(\varepsilon_i \varepsilon_j)$, the universality class is changed from Gaussian to Ising (Lee and Shrock 1987). The second exact result is the 1D $O(N)$ models noted in § 3, which never exhibit criticality or spontaneous symmetry breaking, even at $T = 0$. As these comparisons show, the question of the effect of changing the interaction from a continuously valued to a discretely valued one in a model with continuous parameter space P and zero-field symmetry group G is an interesting and delicate one. Since models with continuous P and G but a discretely valued interaction are realised both in the context of topological density terms in lattice gauge theories and spin systems and in the context of neural networks, it is important to understand the behaviour of such systems.

There are several interesting topics for further investigation. First, as a consequence of the underlying property of greater disorder allowed by the $\mathrm{sgn}(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)$ interaction, as compared with the regular usual $\boldsymbol{S}_i \cdot \boldsymbol{S}_j$ interaction, neither the short-range nor the long-range order saturates at unity in the sgn O(2) model. It would be very enlightening to find an analytic derivation of the $T = 0$ values of the nearest-neighbour spin-spin correlation function and the magnetisation (4.12) and (4.13) which we found by Monte Carlo measurements. Among other topics, it would be valuable to study correlation functions in theories such as the 3D sgn O(2) model. Although universality considerations would lead one to expect that the mass gap exponent ν and the exponent η describing critical correlations would be the same as for the regular 3D O(2) model, the greater floppiness of the spins allowed by the sgn interaction, as compared to the usual spin-spin interaction, should cause the connected correlation functions to fall off more rapidly with the distance between the spins in the sgn model. Third, models such as those investigated in Kohring *et al* (1986) and Kohring and Shrock (1987) with both continuously and discretely valued interactions show a wealth of new features associated with the interplay between the two interactions. It would be worthwhile to study further models of this sort to elucidate their general behaviour.

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